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On $S_R((12))$-blocks for a symmetric group

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1 Main result

Let p be a prime number and let (K, R, k) be a p-modular system,that is,R is a noetherian, complete discrete valuation ring, K the quotient field of R of characteristic zero and k the residue class field of R of characteristic p. For a finite group G, we let $bl(G)$ denote the set of the blocks of RG. If B is a block of RG, then δ_B denotes the identity of B, which is called the block idempotent of B. In this article,we announce the following result:

Theorem A.Let $G = S_n$ be the symmetric group on n letters, $H = \{1, (12)\}$ and $I(H) = RG\sigma_H RG$, where $\sigma_H = 1 + (12)$. If $p > 2$, then $I(H) = \bigoplus_{B\in bl(G)} I(H)\delta_B$ is an indecomposable decomposition of $I(H)$ as a (G, G)-bimodule.

2 Necessary steps

Theorem A is closely related with the notion of the $S_R(H)$-block. Namely for any finite group G and any $H \leq G$, we know that $\mathrm{End}_{G\times G}(I(H))$ is isomorphic to the center of $\mathrm{End}_G(\sigma_H RG)$, where $\sigma_H = \sum_{x\in H} x$ (Robinson [4]). Hence Theorem A is equivalent to:

Theorem A$'$. With the assumptions of Theorem A, we have that $\{\delta_B e_H; B \in bl(G)\}$ is just the set of the block idempotents of the Hecke algebra $e_H RGe_H$, where $e_H = \sigma_H/2$.

We remark that Therem A$'$ (and hence Theorem A) remains true if R is replaced by k. To prove Theorem A,we need a further reduction.For $\chi \in \mathrm{Irr}(G)$, let $\mathrm{IBr}(\chi) = \{\psi \in \mathrm{IBr}(G); d_{\chi\psi} \neq 0\}$. Now Theorem A$'$ is equivalent to :

Theorem A$''$. With the assumptions of Theorem A,let B_0 be the principal block of RG. Then for any $\chi, \chi' \in \operatorname{Irr}(B_0)\backslash\{1_G\}$, there exists a sequence:

$$\chi = \chi_0, \chi_1, \ldots, \chi_s = \chi'$$

of ordinary irreducible characters of G such that $\operatorname{IBr}(\chi_i) \cap \operatorname{IBr}(\chi_{i+1})$ contains an element other than the trivial character for each i $(0 \leq i \leq s-1)$.

Looking at some of the known decomposition matrices of the symmetric groups,one will be convinced that Theoem A$''$ must be true without taking much time.However to prove it is quite another. Now,let S^λ and $\overline{S}^\lambda$ be the Specht modules of G over K and k respectively associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ are positive integers such that $\sum_{i=1}^{r} \lambda_i = n$. Let us write $r = d(\lambda)$. We call the partition λ to be p-regular if there is no i such that $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+p-1}$. Let Λ and Λ_0 be the sets of the partitions and the p-regular ones of n respectively.As is well-known,we have that $\operatorname{Irr}(G) = \{S^\lambda; \lambda \in \Lambda\}$. On the other hand, if $\lambda \in \Lambda_0$, $D^\lambda = hd(\overline{S}^\lambda)$ is simple and $\operatorname{IBr}(G) = \{D^\lambda; \lambda \in \Lambda_0\}$. In particular,we have that $\operatorname{IBr}(S^\lambda\} \ni D^\lambda$ if λ is p-regular, where $\operatorname{IBr}(S^\lambda)$ is the set of irreducible constituents of the kG-module $\overline{S}^\lambda$.For the analysis of the sets $\operatorname{IBr}(S^\lambda)$ we need the theorems of Carter and Payne and of Schaper.To explain the theorem of Carter and Payne,let $[\lambda]$ be the Young diagram associated with the partition λ. We write $\lambda \rightarrow \mu$ if $[\mu]$ can be obtained from $[\lambda]$ by raising q spaces from a row of $[\lambda]$ to a higher row and each space is moved through a multiple of p^e spaces,where $p^e > q$.Then

Theorem B (Carter and Payne [1]). Let λ, $\mu \in \Lambda$. If $\lambda \rightarrow \mu$, then

$$\operatorname{Hom}_G(\overline{S}^\mu, \overline{S}^\lambda) \neq 0.$$

For the theorem of Schaper,see James [3].Using the theorem,we get the following result,which is crucial for the proof of Theorem A$''$.

Lemma C. Let λ be a p-regular partition of n different from (n).If $\operatorname{IBr}(S^\lambda) = \{D^\lambda, 1_G\}$,we have $d(\lambda) = 2$,unless $\lambda = (p-1, p-1, 1)$.

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References

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